Problem 1

Suppose we had 2 solutions u_1, u_2 then $w = u_1 - u_2$ satisfies

3 points for setup

$$\partial_t w = \Delta w \quad x \in \Omega, t > 0$$
$$\partial_\nu w = 0 \quad x \in \partial\Omega.$$
$$w\big|_{t=0} = 0$$

3 points

Now we consider the "energy".

$$E(t) = \int_\Omega (w(t,x))^2 dx$$

2 points

We have that

$$\frac{d}{dt} E(t) = 2 \int_\Omega w \, \partial_t w \, dx$$

8 points

$$= 2 \int_\Omega w \Delta w \, dx$$

Green's thm.

$$= +2\left(-\int_\Omega |\nabla w|^2 dx + \int_{\partial\Omega} w \frac{\partial w}{\partial \nu} dx\right)$$

Since $\partial_\nu w = 0$ on $\partial\Omega$ then

$$\frac{d}{dt} E(t) = -2 \int_\Omega |\nabla w|^2 dx \leq 0.$$

2 points

However if $\nabla w \neq 0$ everywhere then $\frac{dE(t)}{dt} \leq 0$ But this is a contraction sice $E(0) = 0$. giving $E(t) \leq 0$ and $w = 0$.

6 ~~4~~ points to conclusion

Problem 2

We solve the equation via seperation of variables

$$\phi(t,x) = X(x)\,T(t)$$

then

$$\underline{T''(t)}\,X(x) = X''(x)\,T(t)$$

(5 points)

$$\frac{X(x)}{X''(x)} = \frac{T(t)}{T''(t)} = \lambda$$ is a constant

given Dirichlet BC's its good to pick $\lambda = n$

(5 points)

Then

$$\phi_n(t,x) = (A_n \sin nx + B_n \cos nx)$$
$$(\tilde{A}_n \sin nt + \tilde{B}_n \cos nt)$$

Bounday Conditions Indicate

$B_n = 0$

The initial conditions

$$n\hat{A}_n \cos nt - n\tilde{B}_n \sin nt = \frac{\partial T}{\partial t}(t)$$

$$n\hat{A}_n - n\tilde{B}_n \sin(0n) = \frac{\partial T}{\partial t}(0)$$

$\Rightarrow \hat{A}_n = 0$. 3 points

Then

$$\phi_n(t,x) = (A_n \sin nx)(\tilde{B}_n \cos nt)$$

relabelling $A_n \hat{B}_n = C_n$

$$\phi_n(t,x) = C_n(\sin nx)(\cos nt).$$

3 points

It remains to find c_n.

$$\sum_{n} c_n \phi_n(t,x) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

$$= \phi(t,x)$$ 1 point

with $\phi(0,x) = \sin^3 x$

$$\sum_{n=1}^{\infty} c_n \sin nx = \sin^3 x .$$

2points

and c_n is the Fourier sine series coefficients for $\sin^3 x$.

Half Series is

$$\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

here $L=\pi$

$$\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$$

$f(x) = \sin^3 x$ 5 points.

either using angle addition formula or realizing that

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

gives the only coeff. are $n=1$, $n=3$ which are themselves.

The sum of solutions to a linear PDE is also a PDE solution (principle of superposition)

Thus $\phi_1(t,x) = x$ solves

$\phi(0,t) = 0$ $\phi(\pi,t) = \pi$ and

$\frac{\partial \phi_1}{\partial t}(x,0) = 0$ and 4 points

ϕ_2 fond in step (2) solves

$\phi_2(0,x) = \sin^3 x$

$\phi_2(0,t) = 0$ $\phi_2(\pi,t) = 0$

$\frac{\partial \phi_2}{\partial t}(x,0) = 0$ 4 points adding the solutions gives

$\phi_1 + \phi_2$ solves the

wave eq with new BC's

and initial conditions

2 points

Solution is then

$$\phi(t,x) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt + x$$

with c_n found previously.

We write   also OK to use D'alembert + rearrange.

$$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$$

$$\sin nx \cos nt = \frac{\sin(n(x+t))}{2} + \frac{\sin(n(x-t))}{2}$$

$\forall n$

This gives   3 points for each one.

$\phi(t,x) =$

$$\underbrace{\sum_{n=1}^{\infty} \frac{c_n}{2} \sin(n(x+t)) + \frac{x+t}{2}}_{f(x+t)}$$

$$+ \underbrace{\sum_{n=1}^{\infty} \frac{c_n}{2} \sin(n(x-t)) + \frac{x-t}{2}}_{g(x-t)}$$

is the same sol. as before

with $f(x) = \sum_{n=1}^{\infty} \frac{c_n}{2} \sin nx + \frac{x}{2}$

$g(x) = \sum_{n=1}^{\infty} \frac{c_n}{2} \sin nx + \frac{x}{2}$

Problem 3

Solution which satisfies prescribed ic $u(0,x) = f(x)$ is

$u(t,x) = f \circ \beta^{-1}(\beta(x) - t)$ 5 points

$\beta(x) = -\int \frac{dx}{x} = t + k$

$-\ln x = t + k$

3 points

$\beta(x) - t = -\ln x - t$

$\beta^{-1}(x) = e^{-x}$

2 points

Then $\beta^{-1}(\beta(x) - t) = e^{-(-\ln x - t)}$

$= x e^{t}$

$u(t,x) = \frac{1}{(xe^{t})^2 + 1}$

points

Taking the limit we have

$$\lim_{t \to \infty} u(t, x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

solutions must tend to fixed points or blowup.

5 points for conclusion